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Geometric Division with a Fixed Point: Not Half the Cake, But At Least 4/9

ANDREAS WAGENER*
Department of Economics, University of Vienna, Hohenstaufengasse 9, 1010 Vienna, Austria
(E-mail: andreas.wagener@univie.ac.at)

Abstract

We study a two-person problem of cutting a homogeneous cake where one player is disadvantaged from the outset: Unlike under the divide-and-choose rule he may only choose a point on the cake through which the other player will then execute a cut and then take the piece that he prefers. We derive the optimal strategy for the disadvantaged player in this game and a lower bound for the share of the cake that he can maximally obtain: It amounts to one third of the cake whenever the cake is bounded. For convex and bounded cakes the minimum share rises to 4/9 of the cake.

Key words: cake cutting, unfair division

JEL-Classification: D61, D63

1. Introduction

The study of fair-division problems is devoted to finding and implementing procedures to share a desirable or undesirable object among several parties according to some idea of fairness. The *cake-cutting* problem is probably the most familiar example, but problems of allocating burdens (*chore division*) and also mixed, *rent partitioning* problems are widely discussed.[1]

In two-player settings the best-known cake-division method is the *divide-and-choose* rule (Brams and Taylor 1996, p. 8ff.). Player 1 cuts the cake, and Player 2 chooses one of the two pieces. If the cake is perfectly divisible (i.e., if it can be cut at any point without diminishing its value), then divide-and-choose implements a solution that is both proportional (each of the $n = 2$ players feels that he received at least $1/n$ of the cake) and envy-free (no player feels that anybody else got a larger piece than he himself did). An equivalent method is the *moving-knife* or Dubins-Spanier procedure Taylor (1996). Here the cake is supposed to lie entirely above a horizontal axis. A referee holds a knife at the far left of the axis such that the entire cake lies to the right of the knife. He moves the knife in a parallel fashion along the horizontal axis across the cake. At any point in time, any player can call "stop". The player who calls first receives the piece to the left of the knife, the rest being given to the other player. This procedure can easily be extended to n players: Move the knife, and give

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to the first who calls "stop" the piece left to the knife; then continue with the rest of the cake and the remaining $n-1$ players. The outcome of this procedure is still proportional, but no longer envy-free for more than two players: Each of the first $n-2$ players to call "stop" might envy the remaining players if he feels that the rest of the cake is not split evenly between them.[2]

In this paper we discuss an *explicitly* unfair cake- or chore-division situation with two players, i.e., a division problem where one of the two players is disadvantaged from the outset. Situations with unequal positions of two or more players in cake or chore division problems are quite common in reality. Here we phrase the asymmetry between the agents in "geometric" terms. Specifically, we consider the following

Unfair-Division Problem: *A homogeneous cake of form F and size S has to be divided among two individuals, Player 1 and Player 2. Player 1 has the right to fix a point M (on or off the cake, which does not matter). Player 2 then divides the cake with a straight cut through the point M chosen by Player 1. Player 2 chooses one piece of the cake while Player 1 gets the rest.*

What is the share of the cake that Player 1 can maximally get in this game? And how can he achieve that?

This problem came to the author's mind when he observed his landlord's sons argueing about mowing the lawn in the garden, a chore which they were supposed to share. They agreed that one of them (Player 2 in the above description) would mow that area of the lawn that lay "on one side" of an apple tree located somewhere in the middle of the garden; the second brother (Player 1) would then mow the area on "the other side". Obviously, this rule left considerable discretion for Player 2's mowing strategy (i.e., his partitioning of the lawn) – which he unrelentingly exploited to his brother's anger. The Unfair-Division Problem presented above is a stylized and formalized version of this homey lawn-mowing story. It requires the lawn to be partitioned by a straight line and puts the story as a sequential, non-cooperative game.

Geometrically, in the standard cake-cutting or chore-division game Player 1 does not only determine the point through which Player 2's cut has to run, but also the *direction* into which Player 2 has to execute the cut. This is equivalent to a game where Player 1 himself cuts the cake and Player 2 may choose whatever piece he prefers. It is also strategically equivalent to a game where Player 1 determines only the direction of the cut, Player 2 executes the cut, and Player 1 may afterwards choose which piece of the cake to take. These equivalences are due to the fact that for any bounded geometric figure F (which need neither be convex nor consist of a single piece) and *any* point M in the plane there exists a straight line through M that divides F into two pieces of equal area. Similarly, for any bounded geometric figure and any straight line ℓ_1 there exists a line ℓ_2 parallel to ℓ_1 that divides F into two equally large pieces.[3] Hence, Player 1 can always ensure one half of the cake for himself. Furthermore, in the standard cake-cutting game the positions of divider and chooser are equally attractive for both players (in the sense that they have the same maximin payoff).[4]

In the Unfair-Division Problem, Player 1 is disadvantaged, both relative to Player 2 and to the standard cake-cutting problem. We can therefore expect that he will generally obtain less than half the cake. Neither will the solution be envy-free nor are players indifferent between the roles of the first and the second mover. Therefore, our focus cannot be on envy-freeness or proportionality. Rather we ask: To exactly which extent is Player 1 disadvantaged? And what are the best strategies for the two players in the unfair-division problem?

It will turn out that answers to these two questions emerge from some (more-or-less) well-known results from planar geometry. As the answer at least to the first question depends on the geometric shape of the cake, we consider different cases.

2. The Solutions

2.1. Two extreme cases

A trivial case emerges when the cake is centrally symmetric (say, a circle, an ellipse or a rectangle). Then Player 1 should choose the center of symmetry. Any cut through that point will split the cake into areas of equal size. Hence, the positional disadvantage does not involve a loss of cake for Player 1:

Solution 1. *In the Unfair-Division Problem, Player 1 can ensure to obtain one half of the cake if the cake has a centrally symmetric shape.*

Obviously, the symmetric cake is a very special and, from Player 1's perspective, extremely lucky case. At the other extreme (which still has to be proven to be a polar case), consider the following "cake" which consists of three large, non-intersecting circles around the vertices of an arbitrary but, relative to the radius of the circles, sufficiently large triangle. The problem might be the division of an archipelago among two countries:

In a situation like in Figure 1, Player 1 can at most ensure to get 1/3 of the cake, namely by choosing a point like M. Obviously, Player 2 then cannot execute any cut through M

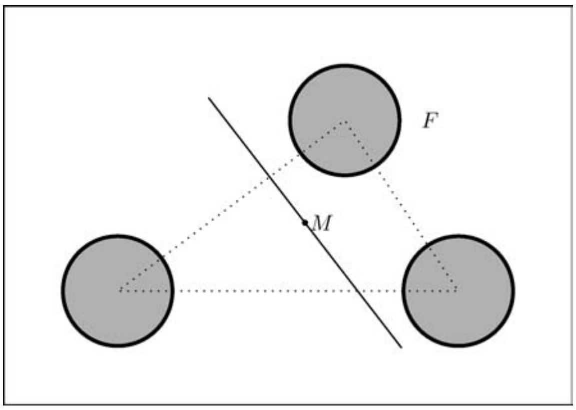

Figure 1. Player 1 gets at least one third of F.

that will earn him more than $2/3$ of the cake (namely, by chopping off two of the circles). One such cut is indicated by the straight line through M.[5]

2.2. Bounded cakes

While the "cake" in Figure 1 does not exactly look like a pie in the common understanding of the word, it represents an interesting problem in its own right as it constitutes the limiting case of the following

Theorem 1 (Yaglom and Boltanskiĭ, 1961, Theorem 2.6c (p. 19, pp. 126f)**.** *For any bounded (but not necessarily convex) figure F of area S there exists a point P such that any straight line through P cuts the figure into two parts whose area is at least $S/3$ each.*

Proof: Consider a bounded but not necessarily convex figure F of area S (as in Figure 2).

Draw a large circle C around F; this is always possible as F is bounded. Consider all halfplanes which contain more than 2/3 of the area of F. The intersection of any such halfplane with C is bounded, convex and, by construction, contains 2/3 of the area of F. Arbitrarily select three of these intersections; call them H_1, H_2, and H_3.

Now consider the intersection of H_1 and H_3. As each of H_1 and H_2 contains at least 2/3 of the area of F and total area of F is S, at least an area of $S/3$ lies inside $H_1 \cap H_2$. As furthermore H_3 contains more than 2/3 of the area of S, at least one point of $H_1 \cap H_2$ must lie in H_3. Hence, any arbitrary three-element selection of intersections of C with halfplanes each of which contains at least 2/3 of the area of F has a non-empty intersection. Now apply *Helly's Theorem*[6] to obtain that the intersection of *all* halfplanes that contain at least 2/3 of the area of F is non-empty.

Let M be an element in that intersection of halfplanes. We are done if we can show that every straight line through M divides F into two parts each of which contains an area of at least $S/3$. Assume the contrary: suppose there exists a straight line ℓ through M such that one of the two halfplanes into which ℓ partitions the plane contains less than $1/3$ of the area of F. For simplicity call this halfplane the *left* one; its complement, the *right* halfplane, then contains more than 2/3 of the area of F. Now draw a parallel ℓ' to the right of ℓ such that the halfplane to the right of ℓ' still contains more than 2/3 of the area of F; such a parallel line always exists by a continuity argument. As more than 2/3 of the area of F lies to the right of ℓ, the point M must also lie in that area (recall that M is in the intersection of *all* halfplanes that contain at least 2/3 of the area of F). But, by construction M is to the left of ℓ' – which establishes a contradiction. Hence, M divides F into parts each of which has an area of more than $S/3$. □

Figure 1 demonstrates that the value $S/3$ given in Theorem 1 is indeed a lower boundary: It depicts a (non-convex) figure F for which no point M can be found such that the ratio of the areas S_1 and S_2 of the two parts F_1 and F_2 into which any line through M divides F can be kept within narrower limits than $0.5 \leq S_1/S_2 \leq 2$. The immediate corollary to this observation is

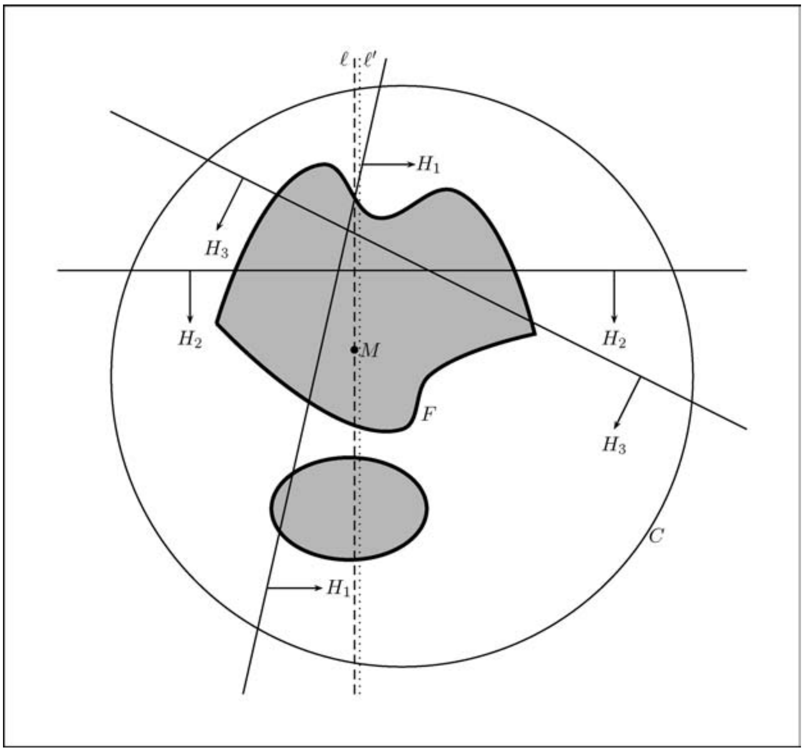

Figure 2. Illustration for the Proof of Theorem 1

Solution 2. *In the Unfair-Division Problem, Player 1 can ensure to obtain at least one third of the cake whenever the cake is bounded.*

2.3. Triangular cakes

Let us now consider the case of a triangular cake. This case will also serve instrumental for the more general discussion of convex cakes below.

Without loss of generality, we consider the case of an equilateral triangle: by an appropriate affine transformation, any triangle can be mapped into an equilateral one (Berger, 1987, ch. 2). The midpoints of the sides, the medians, and the centroid of the original triangle will then be mapped into the midpoints, the medians, and the centroid of the equilateral triangle.

Recall that in any triangle the medians are concurrent in the centroid (center of mass) and that they are each divided by the center of mass in a ratio of $2/3 : 1/3$, measured from the vertices.

Solution 3. *If the cake in the Unfair-Division Problem is triangular, the best strategy for Player 1 is to choose the center of mass of the cake. Player 2 will execute a cut parallel to*

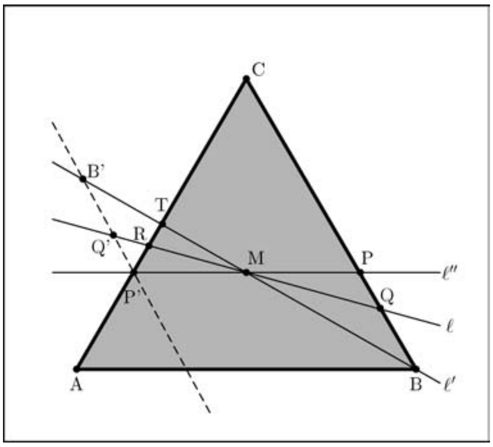

Figure 3. A Triangular Cake

one of the sides of the cake. Player 1 will get exactly 4/9 of the cake, Player 2 the remaining 5/9.

To understand this solution consider Figure 3 where the cake is an equilateral triangle ABC.

Suppose that Player 1 has chosen the center of mass M as the point through which Player 2 has to partition the cake. What should Player 2 do? By cutting along the median (line ℓ' through B and T, the midpoint of AC), the triangle will be exactly halved. Now rotate ℓ' to, say, ℓ which crosses the edges AC and BC at points R and Q, respectively.[7] This will increase the size of that part of the cake that is below the cut by the area of the quadrilateral $RTB'Q'$, where B' and Q' are the symmetric images of B and Q with respect to M; observe that the quadrilateral $RTB'Q'$ then has the same area as the difference between the areas of the triangles MBQ and RMT. Extending the argument one sees that the area below the cut will be maximized by the line ℓ'' that parallels AB through point M. The area of the trapezoid $ABPP'$ equals, by construction, 5/9 of the area of the triangle (the triangle above PP' contains 4/9 of the area of ABC since both its base line and its altitude are, in length, 2/3 of the corresponding lines in the original triangle).

Now consider why it is an optimal strategy for Player 1 to select M as the point through which Player 2 has to cut the cake: Any point other than M would give Player 2 the opportunity to draw a cut parallel to that side of the triangle that has the largest distance from the point chosen by Player 1. Player 2 would walk away with a trapezoid which has, when compared to trapezoids that can be cut off by lines through M, the same base line $AB = BC = AC$, but a greater altitude than $ABPP'$ – and consequently a larger size. Evidently, this would harm Player 1, and Solution 3 indeed describes the subgame-perfect equilibrium of the Unfair-Division Problem.

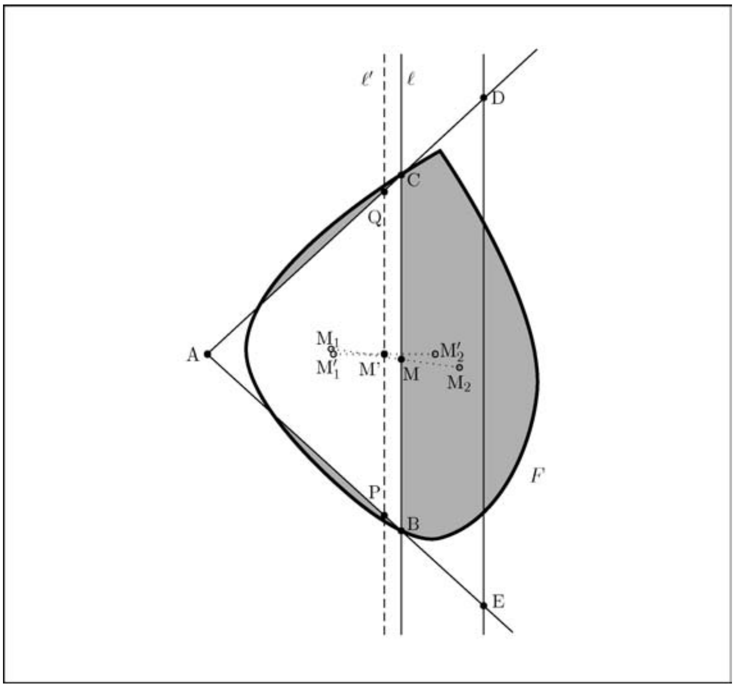

Figure 4. Player 1 gets at least 4/9 of F.

2.4. Bounded convex cakes

The following theorem, attributed to Winternitz by Yaglom and Boltjanskiĭ (1961, p. 36), gives the general answer to the problem of cutting bounded convex cakes with a rotating-knife procedure imposed in the Unfair-Division Problem:

Theorem 2 (Winternitz)**.** *For any bounded and convex figure F of area S there exists a point M (namely, the center of mass) such that any straight line through M cuts the figure into parts whose area is at least $4/9 \cdot S$ each.*

Proof: The proof is adapted from Yaglom and Boltjanskii (1961 pp. 160–162). Let a convex and bounded figure F with area S be given (for an illustration see Figure 4). Denote the center of mass of F by M. We cut F by straight lines through M into two parts and compare the ratios of the areas of the resulting parts of F. It is shown that one can obtain a triangle T for which the ratio of the areas that result from cutting F through M is smaller than for the figure F itself (see (1) below).

Step 1: Let ℓ be an arbitrary line through M. It partitions F into two parts, say, F_1 and F_2 with areas S_1 and S_2. In Figure 4, let F_1 be the part of F to the left of ℓ. Replace F_1 by a triangle ABC such that (i) B and C are the points of intersection of ℓ with the closure of F and (ii) the triangle has an area equal to S_1. Choose A to lie in the interior of that region that is formed by ℓ and the two supporting lines to F through B and C.[8]

Next replace F_2 by a trapezoid $EDCB$ such that (i) the segment BC is the smaller base, (ii) points E and D lie on the straight lines AB and AC, respectively, and (iii) the area of $EDCB$ equals S_2. By construction, F has the same size as the triangle AED.

Step 2: Let M' be the center of mass of AED. We will show that M' belongs to ABC. To that end, let M_1 and M_2 be the centers of mass of F_1 and F_2 and let M_1' and M_2' be the centers of mass of the triangle ABC and the trapezoid $EDCB$, respectively.

M is on the line M_1M_2 and M' is on $M_1'M_2'$. The point M divides M_1M_2 by ratio $M_1M : MM_2 = S_1 : S_2$. So does M' with $M_1'M_2'$. Now verify that M_1' is not closer to ℓ than M_1; on the other hand, M_2 is not closer to ℓ than M_2'. Hence, M' is on the same side of ℓ as A.

Step 3: Draw a parallel ℓ' to ℓ through M'. Denote the intersections of ℓ' with the sides AB and AC of the triangle by P and Q. By construction,

$$\text{area}(\triangle_{APQ}) \leq \text{area}(\triangle_{ABC}) = S_1 \quad \text{and} \quad \text{area}(\square_{EDQP}) \geq \text{area}(\square_{EDCB}) = S_2,$$

which implies

$$\frac{S_1}{S_2} \geq \frac{\text{area}(\triangle_{APQ})}{\text{area}(\square_{EDQP})}. \tag{1}$$

Step 4: We will show that

$$\frac{\text{area}(\triangle_{APQ})}{\text{area}(\square_{EDQP})} = \frac{4}{5}. \tag{2}$$

To see this, recall that the triangle ABC is similar to the triangle AED. Since the medians (which intersect in the center of mass) are each divided by the center of mass in a ratio of 2/3:1/3, the ratio of the areas of the triangles ABC and AED is 4:9 or, as AED has area S,

$$\text{area}(\triangle_{APQ}) = \frac{4}{9} \cdot S \quad \text{and} \quad \text{area}(\square_{EDQP}) = \frac{5}{9} \cdot S.$$

Step 5: Combining (1) and (2), the ratio of the areas of the parts into which any convex and bounded figure F is divided by an arbitrary cut through its center of mass can never exceed 4:5. I.e., the area of the smaller part will never fall below $4/9$ of the area of F – which was to be shown. It will exactly equal $4S/9$ iff F is a triangle. □

An alternative formulation of Winternitz' Theorem[9] is as follows: For any convex planar body F with area S and any halfplane H that contains the center of mass of F, the area of $H \cap F$ is at least $4S/9$. An immediate corollary to this theorem is

Solution 4. *In the Unfair-Division Problem, Player 1 can always ensure to obtain at least 4/9 of the cake whenever the cake has a bounded and convex shape. To achieve that, he chooses the center of mass of the cake as the point through which player 2 partitions the cake.*

3. Conclusion

We set up a two-player cake-cutting problem with a rotating-knife procedure where one player is *a priori* disadvantaged. This gives rise to a sequential game with a first-mover disadvantage. One can establish precise upper bounds for the losses which the disadvantaged player has to incur relative to the standard fair-division setting. With convex-shaped cakes, the maximum loss is surprisingly small: only 1/18 of the cake. From the viewpoint of Player 1, the "worst-cake" scenario is the triangular cake, the best scenario is the centrally symmetric cake.

Unlike many other cake- or chore-division scenarios, the problem presented here is entirely geometric: The only interesting feature of the cake is its shape. On one hand, the geometric analysis allows for a remarkable precision in the solution of the problem. On the other hand, the geometry also limits applicability of the results to more general settings. Interesting extensions of Unfair-Division Problems would encompass settings with more than two player, with heterogeneous cakes, or with cuts other than straight lines.

A potential and important area of application of our analysis are *location games* (for a survey see Eiselt et al., 1993): Consider a two-dimensional region ("market") that is uniformly populated by consumers. Two salesmen 1 and 2 of a homogeneous good sequentially choose where to establish a shop in the market; salesman 1 moves first. After shops have opened, each consumer will buy one unit of the good from the shop closest to his residence. Each firm is supposed to maximize its market share. Under the natural assumption that with equal locations the market will be shared evenly, salesman 1 can never earn a market share of more than 50% in a subgame-perfect equilibrium of this game. Unfortunately, however, not much more is known on the equilibria of such games. Observe now that the geometric structure of this location game coincides with that in the Unfair-Division Problem: The region corresponds to the cake, and market shares to the fractions of the cake allocated to players (=salesmen) 1 and 2. Any two non-identical shop locations uniquely determine a straight line that partitions the market.[10] A conjecture is that the location game and the Unfair-Division Problem are strategically equivalent in the sense that the payoffs and the market partition resulting in a subgame-perfect equilibrium of a game where players sequentially choose locations are identical to one where Player 1 chooses a boundary point of its market region before Player 2 then draws a straight separation line running through that boundary point. Indicative for that conjecture is a recent result by Chawla et al. (2003): The worst case for Player 1 over all two-dimensional market shapes in a location game is as depicted in Figure 1; it results in a market share of 1/3 for Player 1. This result is very much reminiscent of Solution 2 above. Yet, proving the conjecture that one-shot sequential location games and the Unfair-Division problem are fully equivalent is an open question for future research.

Notes

1. Comprehensive surveys as well as plenty of applications are provided by Brams and Taylor (1996), Young (1994) and Robertson and Webb (1998).
2. Brams and Taylor (1995) provide a constructive algorithm that achieves an envy-free division of a cake among n players.
3. Consider a geometric figure F with area S, a horizontal axis h, and a vertical straight line ℓ. The area $S(x)$ of the part of the figure located to the left of ℓ is a continuous function of the coordinate x of the intersection of ℓ and the horizontal axis h. Since F is bounded one can always find lines ℓ_1 and ℓ_2 and corresponding values x_1 and x_2 for the coordinates of intersection with h such that $S(x_1) = 0$ and $S(x_2) = S$. Hence, by the intermediate value theorem, there exists x_0 such that $S(x_0) = S/2$. A similar argument applies to the size y of the angle that any straight line ℓ forms with the origin of a fixed axis that does not intersect with F: The areas of the parts of F that are located in either of the halfplanes created by ℓ are continuous functions of y.

   Brams and Taylor (1996) discuss the moving-knife procedure for cakes of rectangular shape. As this argument above shows, the method also works for arbitrary bounded (and not necessarily convex-shaped) cakes.
4. Divider's and chooser's positions are not equally attractive if there exists a division of the cake that Pareto-dominates the equal-split solution (see Young 1994, pp. 137ff). As we only consider a single, homogeneous and perfectly divisible cake and players' preferences that are monotonic in the same direction with respect to the cake's size, this problem is not relevant here.
5. If the triangle is smaller (relative to the radius of the circles) no point exists such that a straight line through it will separate two of the circles from the third. Consequently, Player 1's share of the cake will be larger than 1/3.
6. See Berger (1987, Section 11.7). Helly's Theorem states that, if in a collection of (finitely or infinitely many) bounded and convex figures each triple has a common point, then a point exists which simultaneously belongs to all figures in the collection. In the proof, the circle C ensures that the H_i are bounded – which is sufficient to make Helly's Theorem applicable for an infinite number of intersecting sets.
7. By reasons of symmetry it does not matter whether we rotate ℓ' upwards or downwards.
8. These supporting lines are not drawn in Figure 4, but it is easy to see that, in the present case, they intersect to the left of ℓ. Should they intersect to the right of ℓ choosing A is even simpler; the proof is not affected.
9. Grünbaum (1960) generalized Theorem 2 to more than two dimensions: *Let K be a bounded and convex body in the Euclidean n-space and $V(K)$ its volume. Any hyperplane through the center of mass of K partitions K into two subsets K_1 and K_2 whose volumes satisfy*

$$\frac{V(K_i)}{V(K)} \geq \left(\frac{n}{n+1}\right)^n \quad (i = 1, 2).$$

   Hence, with three-dimensional cakes Player 1 could get at least 27/64 = 42.2% of the cake's volume in an Unfair-Division-Problem. Since $((n+1)/n)^n < e$ for all n, any hyperplane through the centroid cuts out at least a share of $1/e$ of the mass of K.

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