Measures of Risk Attitudes:
Correspondences between Mean-Variance- and Expected Utility Approach*

Thomas Eichner
Department of Economics, University of Siegen

Andreas Wagener†
Department of Economics, University of Vienna

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† *Corresponding author.* University of Vienna, Department of Economics, Hohenstaufengasse 9, 1010 Vienna, Austria. Phone: +43 - 1 - 4277 37423. E-mail: andreas.wagener@univie.ac.at.

1. Introduction

The expected-utility (EU) approach entails great richness to deal with decision making in a large variety of stochastic environments. Research in the EU-paradigm often starts from plausible assumptions on risk preferences or optimal responses to changes in the risk structure, and then investigates how such assumptions are reflected by properties of the von-Neumann-Morgenstern (vNM) utility functions underlying the EU concept. Building on Pratt (1964)'s analysis of risk aversion, several measures for risk attitudes have been analyzed, including absolute prudence and temperance (Kimball, 1990; Eeckhoudt et al., 1996), their relative and partial relative counterparts (Choi et al., 2001; Honda, 1985) as well as extensions such as mixed risk aversion (Caballé and Pomansky, 1996).

The two-parameter, mean-variance specification of risk preferences certainly counts among the most popular approaches towards decision making under uncertainty within both economic theory and practical applications. In this approach, utility from a lottery is represented by a function (only) of the first and second moments of the distribution of payoffs (income, say). Since its earliest days (Markowitz, 1959, pp. 286-288), (μ, σ)-analysis has always been recognized as deficient, relative to the EU approach: It is only consistent with the EU approach for quadratic utility functions (Baron, 1977) or if all random variables are jointly elliptically distributed (Chamberlain, 1983). As shown by Owen and Rabinovitch (1983), the latter condition holds if all attainable distributions differ only by location and scale parameters — which, as Meyer (1987) and Sinn (1983) argue, covers a wide range of economic decision problems.

When (μ, σ)- and EU-approach are perfect substitutes, a number of formal correspondences between them can be identified. Meyer (1987) translates the measures of absolute and relative risk aversion and their monotonicity properties from the EU approach into the two-parameter framework. Lajeri and Nielsen (2000) and Eichner and Wagener (2003a) derive a (μ, σ)-equivalent for Kimball (1990)'s notion of decreasing absolute prudence. Yet, the two-parameter approach is still lagging behind the progress made in the EU-framework.[1] This paper tries to narrow the gap.

For all absolute measures of risk attitudes used in the EU framework we define (μ, σ)-analogues in the form of marginal rates of substitution between μ and σ for the two-parameter utility function or its derivatives (Section 3). We show that the monotonicity properties of these (μ, σ)-measures with respect to μ and σ coincide with the monotonicity properties of, respectively, the corresponding absolute EU-measure and of the corresponding partial relative EU-measure (Section 4).

In Section 5 we discuss the case of normally distributed stochastics. We present slightly different (μ, σ)-measures for risk attitudes and show that they are equivalent to our previous ones for Gaussian stochastics. This finding has an interesting repercussion to the EU-approach: There, indirect (or derived) utility functions *sensu* Kihlstrom et al. (1981) inherit important properties such as decreasing absolute risk aversion or prudence from the original utility index, while the converse is generally not true. Our results show that the converse does indeed only hold for Gaussian stochastics.

2. Notation and preliminaries

Following Meyer (1987), we consider a choice set $\mathcal{Y}$ of random variables (lotteries) that have support in a (possibly unbounded) interval $\mathbf{Y}$ of the real line and that only differ from one another by location and scale parameters. I.e., if X is the random variable obtained by normalization of an arbitrary element of $\mathcal{Y}$, then any $Y \in \mathcal{Y}$ is equal in distribution to $\mu_y + \sigma_y X$, where μ_y and σ_y are the mean and the standard deviation of the respective Y. By $\mathbf{M} := \{(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ | \exists Y \in \mathcal{Y} : (\mu_Y, \sigma_Y) = (\mu, \sigma)\}$ we denote the set of all possible (μ, σ)-pairs that can be obtained for $Y \in \mathcal{Y}$. We assume that $\mathbf{M}$ is a convex set.

[1] To be sure, as the two-parameter approach is less general than the EU-approach, it cannot deal with all the questions amenable to the EU-framework.

Let $u : \mathbb{R} \to \mathbb{R}$ be a vNM utility index; for simplicity we assume that u is a smooth function. Then the expected utility from the lottery Y can be written in terms of the mean and the standard deviation of Y as:

$$\mathbf{E}u(y) = \int_{\mathbf{Y}} u(\mu_Y + \sigma_Y x)\mathrm{d}F(x) =: U(\mu_Y, \sigma_Y) \tag{1}$$

where F is the distribution function of X. Recall that the mean and the standard deviation of X are, respectively, zero and one by construction.

It is evident from (1) that $u(y)$ is increasing for all $y \in \mathbf{Y}$ if and only if $U(\mu, \sigma)$ is increasing in μ for all $(\mu, \sigma) \in \mathbf{M}$. Furthermore, the following relationships hold for all $n \in \mathbb{N}$:[2,3]

$$u^{(n+1)}(y) \lesseqgtr 0 \ \forall y \in \mathbf{Y}$$

$$\iff \quad \frac{\partial^{\,n+1} U(\mu, \sigma)}{\partial\, \mu^{n+1}} \lesseqgtr 0 \quad \forall(\mu, \sigma) \in \mathbf{M} \tag{2a}$$

$$\iff \quad \frac{\partial^{\,n} U(\mu, \sigma)}{\partial\, \sigma \partial\, \mu^{n-1}} \lesseqgtr 0 \quad \forall(\mu, \sigma) \in \mathbf{M} \tag{2b}$$

$$\iff \quad \frac{\partial^{\,n+1} U(\mu, \sigma)}{\partial\, \mu^{n+1}} \cdot \frac{\partial^{\,n+1} U(\mu, \sigma)}{\partial\, \sigma^2 \partial\, \mu^{n-1}} - \left(\frac{\partial^{\,n+1} U(\mu, \sigma)}{\partial\, \sigma \partial\, \mu^{n}} \right)^2 \gtreqless 0 \quad \forall(\mu, \sigma) \in \mathbf{M}. \tag{2c}$$

From (2a), the monotonicity properties of U with respect to μ are reflected by the monotonicity properties of u with respect to y. Analoguous equivalences exist for U_μ and u', and so forth. Eq. (2b) shows that the sign of $u^{(n)}$ is identical to that of the $(n-1)$-st derivative of U_σ with respect to μ. This will play an important role below. Finally, (2c) identifes the curvature properties of $\partial^{\,n-1} U / \partial\, \mu^{n-1}$ as being determined by the curvature of $u^{(n-1)}$ (i.e., the monotonicity of $u^{(n+1)}$). For $n = 1$ this appears already in Meyer (1987) who shows that $U(\mu, \sigma)$ is concave iff $u''(y) < 0$. Meyer's proof for that case can be readily adapted to confirm (2c) for larger n.

3. Measures of risk attitudes

Given a vNM-function utility index u, a prominent class of absolute measures of risk attitudes in the EU-approach is defined by

$$B_n(y) := -\frac{u^{(n+1)}(y)}{u^{(n)}(y)} \tag{3}$$

($y \in \mathbb{R}$, $n \in \mathbb{N}$). $B_1(y)$ is the Arrow-Pratt measure of absolute risk aversion, while $B_2(y)$ and $B_3(y)$ are, respectively, the measure of absolute prudence due to Kimball (1990) and the measure of absolute temperance introduced by Eeckhoudt et al. (1996). The still nameless B_n of higher-order play an important role, e.g., in the concept of *mixed risk aversion* (Caballé and Pomansky, 1996) which in turn has interesting implications for comparative statics. For $y, z \in \mathbb{R}$ and $n \in \mathbb{N}$ further let

$$P_n(y, z) := -z \cdot \frac{u^{(n+1)}(y + z)}{u^{(n)}(y + z)}. \tag{4}$$

[2] For integers $n \in \mathbb{N}_0$, $f^{(n)}(x)$ denotes the n-th order derivative of $f(x)$; by convention $f^{(0)}(x) \equiv f(x)$. In multi-variable functions, subscripts denote partial derivatives.

[3] To obtain equivalence (2b), note that by definition and integration by parts we get

$$\frac{\partial^{\,n} U(\mu, \sigma)}{\partial\, \sigma \partial\, \mu^{n-1}} = \int_{-\infty}^{\infty} x u^{(n)}(\mu + \sigma x)\mathrm{d}F(x) = -\int_{-\infty}^{\infty} \left(u^{(n+1)}(\mu + \sigma x) \int_{-\infty}^{x} z\mathrm{d}F(z) \right) \mathrm{d}x,$$

where the inner integral is always negative due to $\mathbf{E}x = 0$.

For $n = 1$ this yields partial relative risk aversion as introduced by Menezes and Hanson (1970). Cheng et al. (1987) employ properties of P_1 to characterize comparative static results in the EU-framework. We identify $P_2(y, z)$ and $P_3(y, z)$ as, respectively, partial relative prudence and partial relative temperance, which also play important roles in comparative statics; see Choi et al. (2001) for P_2 and Honda (1985) for P_3. Let us further define

$$\beta_n(\mu, \sigma) := -\frac{\partial^n U(\mu, \sigma)}{\partial \sigma \partial \mu^{n-1}} \Big/ \frac{\partial^n U(\mu, \sigma)}{\partial \mu^n}. \tag{5}$$

β_1 is the marginal rate of substitution between μ and σ for utility function U. Meyer (1987, Property 5) shows that this MRS is the two-parameter equivalent of the Arrow-Pratt measure B_1 of absolute risk aversion. For higher values of n, similar analogies will be given in Proposition 1 below.

4. The monotonicity properties of B_n and β_n

Proposition 1. *For all $n \in \mathbb{N}$:*

$$B_n'(y) \underset{<}{\overset{>}{=}} 0 \quad \forall y \in \mathbf{Y} \quad \Longleftrightarrow \quad \frac{\partial \beta_n(\mu, \sigma)}{\partial \mu} \underset{<}{\overset{>}{=}} 0 \quad \forall (\mu, \sigma) \in \mathbf{M}; \tag{6a}$$

$$\frac{\partial P_n(y, z)}{\partial z} \geq 0 \quad \forall y \in \mathbf{Y} \quad \Longleftrightarrow \quad \frac{\partial \beta_n(\mu, \sigma)}{\partial \sigma} \geq 0 \quad \forall (\mu, \sigma) \in \mathbf{M}. \tag{6b}$$

Proof: We only show (6b) since (6a) can be proved by the same technique. Proofs of (6a) for $n = 1, 2$ can be found in Meyer (1987) and Lajeri and Nielsen (2000). Calculate:

$$\begin{aligned}
& \frac{\partial \beta_n}{\partial \sigma} \geq 0 \quad \forall (\mu, \sigma) \\
\Longleftrightarrow \quad & \frac{\partial^{(n+1)} U}{\partial \sigma^2 \partial \mu^{(n-1)}} \cdot \frac{\partial^n U}{\partial \mu^n} - \frac{\partial^{(n+1)} U}{\partial \sigma \partial \mu^n} \cdot \frac{\partial^n U}{\partial \sigma \partial \mu^{(n-1)}} \leq 0 \quad \forall (\mu, \sigma) \\
\Longleftrightarrow \quad & \int x^2 u^{(n+1)} \mathrm{d}F(x) \int u^{(n)} \mathrm{d}F(x) - \int x u^{(n+1)} \mathrm{d}F(x) \int x u^{(n)} \mathrm{d}F(x) \leq 0 \, \forall (\mu, \sigma) \\
\Longleftrightarrow \quad & \int \sigma x^2 \frac{u^{(n+1)} u^{(n)}}{u^{(n)} \int u^{(n)} \mathrm{d}F} \mathrm{d}F - \int \sigma x \frac{u^{(n+1)} u^{(n)}}{u^{(n)} \int u^{(n)} \mathrm{d}F} \mathrm{d}F \int x \frac{u^{(n)}}{\int u^{(n)} \mathrm{d}F} \mathrm{d}F \leq 0 \, \forall (\mu, \sigma) \\
\Longleftrightarrow \quad & -\mathbf{E}_G(x \cdot P_n(\mu, \sigma x)) + \mathbf{E}_G P_n(\mu, \sigma x) \cdot \mathbf{E}_G x \leq 0 \quad \forall (\mu, \sigma) \\
\Longleftrightarrow \quad & \mathrm{Cov}_G(x, P_n(\mu, \sigma x)) \geq 0 \quad \forall (\mu, \sigma) \\
\Longleftrightarrow \quad & \frac{\partial P_n(\mu, z)}{\partial z} \geq 0 \quad \forall \mu, z
\end{aligned}$$

where we set $z \equiv \sigma x$ and the argument of u is always $(\mu + \sigma x)$. The first of these equivalences comes from differentiating β_n. In the second we used (1). The third follows from premultiplying with $\sigma / (\int u^{(n)} \mathrm{d}F)^2 > 0$. To obtain the fourth, we used the distribution function G defined by $\mathrm{d}G = (u^{(n)} / \int u^{(n)} \mathrm{d}F) \mathrm{d}F$; $\mathbf{E}_G$ denotes the expectation operator with respect to G. The fifth equivalence is by definition, the final one is due to Chebyshev's inequality. ■

Both parts of Proposition 1 have interesting implications. We first comment on (6a):

Remark 1. Eq. (6a), combined with (2a) and (2b), states that β_n can be used in the two-parameter approach as the analogue for B_n in the EU approach: The measures coincide both in sign and in monotonicity properties. Hence, β_1, β_2, and β_3 reflect and, conversely, are reflected by absolute risk aversion B_1, absolute prudence B_2, and absolute temperance B_3, respectively. Decreasing absolute risk aversion [prudence, temperance] is mirrored by β_1 [β_2, β_3] being decreasing in μ. For $n = 1$, (6a) already appears in Meyer (1987) who shows that DARA of u has its counterpart in the MRS between μ and σ being decreasing in μ in the two-parameter approach. Lajeri and Nielsen (2000) and Eichner and Wagener (2003a) extend this result to prudence ($n = 2$).

Remark 2. For the EU framework, Kimball (1990) has shown that preferences exhibit DARA if and only if the Arrow-Pratt measure of risk aversion B_1 exceeds the measure of prudence B_2. The following result shows that this observation straightforwardly extends to the (μ, σ)-approach and to measures of higher order:

Corollary 1. *Suppose that $B_n(y) > 0$. Then*

$$B_n(y) \leq B_{n+1}(y) \quad \forall y \in \mathbf{Y} \quad \Longleftrightarrow \quad \beta_n(\mu,\sigma) \leq \beta_{n+1}(\mu,\sigma) \quad \forall (\mu,\sigma) \in \mathbf{M}. \tag{7}$$

Proof: The RHS of (6a) can be written as $\partial \beta_n / \partial \mu = -\left(\frac{\partial^{n+1} U}{\partial \mu^{n+1}} / \frac{\partial^n U}{\partial \mu^n}\right) \cdot (\beta_n - \beta_{n+1})$. Calculate that $B_n'(y) = B_n(y) \cdot (B_n(y) - B_{n+1}(y))$. Now utilize (2a) and (6a). ■

Thus, assertions such as "Risk aversion decreases if and only if risk aversion exceeds prudence" and their kindred also make sense in the two-parameter approach.

Remark 3. Following Caballé and Pomansky (1996), a smooth utility function u on $\mathbb{R}_+$ is said to exhibit *mixed risk aversion* iff its derivatives alternate in sign:

$$(-1)^n \cdot u^{(n)}(y) \leq 0 \quad \forall\, y > 0, \forall\, n \in \mathbb{N}. \tag{8}$$

Using (2a), mixed risk aversion (8) has a straightforward analogue in the two-parameter approach, namely $(-1)^n \cdot (\partial^n U / \partial \mu^n) \leq 0$ for all $(\mu, \sigma) \in \mathbf{M}$ and all $n \in \mathbb{N}$. Caballé and Pomansky (1996, Proposition 3.2) show that (8) is equivalent to all risk measures B_n being non-increasing:

$$B_n'(y) \leq 0 \quad \forall\, y > 0, \forall\, n \in \mathbb{N}. \tag{9}$$

Invoking Proposition 1 and the equivalence between (8) and (9) we obtain a further two-parameter analogue for mixed risk aversion:

$$\frac{\partial \beta_n(\mu,\sigma)}{\partial \mu} \leq 0 \quad \forall (\mu,\sigma) \in \mathbf{M}, \forall n \in \mathbb{N}. \tag{10}$$

Let us now comment on the second part of Proposition 1: Eq. (6b) reveals an equivalence between the monotonicity of measures of partial relative risk measures P_n and the monotonicity of β_n with respect to the standard deviation.

Remark 4. Using their definitions, measures B_n and P_n are related by:

$$\frac{\partial P_n(\mu, x)}{\partial x} \quad = \quad B_n(\mu + x) + x \cdot B_n'(\mu + x).$$

Clearly, $B_n' \leq 0$ is sufficient for $\partial P_n / \partial x > 0$ if $x \leq 0$. Moreover, Hanson and Menezes (1968, Propositions 1 and 2) show that if one wants P_n to be monotone in x for all $x > 0$, then this is only compatible with $B_n > 0$ if P_n is strictly monotonically *increasing*: $\partial P_n / \partial x > 0$.[4] Combined with (6b) this implies that if we want β_n to react uniformly upon changes in σ we are bound to the case $\partial \beta_n / \partial \sigma > 0$.

[4]Hanson and Menezes (1968) only deal with partial relative risk aversion P_1, but their procedure can be extended to P_n for all $n \in \mathbb{N}$.

Remark 5. As is stressed by Hanson and Menezes (1968) and Menezes and Hanson (1970), fluctuations in the sign of $\partial P_n/\partial x$ cannot be excluded generally, and monotonicity of $\partial P_n/\partial x$ thus is an assumption on its own rather than an artefact of the more basic properties of $u^{(n-1)}$. Interestingly, this is different for the sign of $\partial \beta_n/\partial \sigma$. One easily shows that $\partial \beta_n/\partial \mu < 0$ and the curvature properties of $\partial U^{(n-1)}/\partial \mu^{(n-1)}$, see (2c), together imply $\partial \beta_n/\partial \sigma > 0$.[5] Hence, employing (2b) and (6a) one finds

Corollary 2. *For all $n \in \mathbb{N}$:*

$$B_n(y) > 0 \quad \wedge \quad B_n'(y) \le 0 \quad \forall y \in \mathbf{Y} \quad \Longrightarrow \quad \frac{\partial \beta_n(\mu,\sigma)}{\partial \sigma} \ge 0 \quad \forall (\mu,\sigma) \in \mathbf{M}. \tag{11}$$

E.g., decreasing absolute risk aversion implies that the MRS between μ and σ in U is increasing in σ (set $n = 1$). The converse is not true. In applications of the two-parameter approach, the sign of $\partial \beta_1/\partial \sigma$ determines the comparative static effects of optimal risk-taking with respect to increases in risks (see, e.g., Wagener, 2003) where DARA turns out to be a sufficient (but not a necessary) condition such that higher riskiness leads to less risk-taking.
Purely in (μ,σ)-terms, (11) can be written as follows:

$$\beta_n(\mu,\sigma) > 0 \ge \frac{\partial \beta_n(\mu,\sigma)}{\partial \mu} \quad \forall (\mu,\sigma) \in \mathbf{M} \quad \Longrightarrow \quad \frac{\partial \beta_n(\mu,\sigma)}{\partial \sigma} \ge 0 \quad \forall (\mu,\sigma) \in \mathbf{M},$$

while the converse does not necessarily hold. For $n = 1, 2$, this statement has been established by Lajeri-Chaherli (2003, Proposition 3 and 6). Combined with the equivalence between (8) and (9), we then obtain that mixed risk aversion (i.e., $(-1)^n \cdot (\partial^n U/\partial \mu^n) \le 0$ for all (μ,σ) and all n) implies that

$$\frac{\partial \beta_n(\mu,\sigma)}{\partial \sigma} > 0 \quad \forall (\mu,\sigma) \in \mathbf{M} \quad \text{and} \quad \forall n \in \mathbb{N}.$$

Remark 6. The previous remarks imply that $\partial \beta_n/\partial \sigma < 0$ can at most be a local property. Reversing the inequalities in the proof of (6b) yields that, if $\partial \beta_n/\partial \sigma < 0$ were to hold globally, $\partial P_n/\partial x$ would have to be globally decreasing — which is incompatible with $B_n(y) > 0$. An example that $\partial \beta_1/\partial \sigma < 0$ is indeed feasible *locally* is provided in Eichner (2000).

Remark 7. In the EU-framework, measures of risk attitudes B_n and P_n and their properties are useful tools in the analysis of comparative static problems. Quite many of the properties of B_n and P_n represent utility-theoretic equivalents for certain behavioural responses towards changes in stochastic or non-stochastic components of the choice problem at hand (see Gollier, 2000, for a survey). Given the formal equivalences the (μ,σ)-measures β_n and their EU-counterparts B_n or P_n one should expect that the β_n inherit at least parts[6] of their EU-counterparts' behavioural implications as well. Some studies have shown that this is indeed the case:

- Hawawini (1978) graphically shows that DARA in the (μ,σ)-sense of $\partial \beta_1/\partial \mu < 0$ implies that wealthier people are willing to accept absolutely higher risks — which is the original idea of DARA (i.e., $B_1' < 0$) in Pratt (1964).

[5] $\partial \beta_n/\partial \mu < 0$ means $\frac{\partial^{n+1} U}{\partial \sigma \partial \mu^n} \cdot \frac{\partial^n U}{\partial \mu^n} > \frac{\partial^n U}{\partial \sigma \partial \mu^{n-1}} \cdot \frac{\partial^{n+1} U}{\partial \mu^{n+1}}$. Suppose first that $\frac{\partial^{n+1} U}{\partial \mu^{n+1}} \ge 0$. Then (2c) implies that $\partial^{n-1} U/\partial \mu^{n-1}$ is concave, meaning that $\frac{\partial^{n+1} U}{\partial \sigma^2 \partial \mu^{n-1}} \cdot \frac{\partial^{n+1} U}{\partial \mu^{n+1}} \ge \left(\frac{\partial^{n+1} U}{\partial \sigma \partial \mu^n}\right)^2$. In this condition replace one of the $\frac{\partial^{n+1} U}{\partial \sigma \partial \mu^n}$ by the expression emerging from $\partial \beta_n/\partial \mu < 0$ above to obtain that $\partial \beta_n/\partial \sigma > 0$. The case $\frac{\partial^{n+1} U}{\partial \mu^{n+1}} \le 0$ follows along identical lines, taking into account that inequality signs reverse and that $\partial^{n-1} U/\partial \mu^{n-1}$ is convex now.

[6] Recall that we confine ourselves to choice problems where all random variables only differ from one another by location and scale parameters. While innocuous for many economic settings, this restriction reduces comparability between (μ,σ)- and EU-approach to a strict subset of the problems that can be meaningfully analysed in the EU-framework.

- Wagener (2002) shows that absolute prudence in the (μ,σ)-sense of $\beta_2 > 0$ implies the existence of a precautionary motive for saving and that decreasing absolute prudence (i.e., $\partial\beta_2/\partial\mu < 0$) leads wealthier people to cut back their precautionary saving — which corresponds to Kimball (1990)'s original concepts of prudence and decreasing absolute prudence in the EU-context.

This brief list is far from completed yet (see also Remark 8 below). Our quite general findings in Proposition 1 might be instrumental to establish further behavioural equivalences between the two set-ups.[7]

5. Normal distribution

We now turn to the special case that stochastics are Gaussian: $X \sim N(0,1)$. Given the location-scale framework, all $Y \in \mathcal{Y}$ are then normally distributed, too. We find[8]

Proposition 2.

$$X \sim N(0,1) \quad \iff \quad \frac{\partial^n U(\mu,\sigma)}{\partial\sigma\partial\mu^{n-1}} = \sigma \cdot \frac{\partial^{n+1} U(\mu,\sigma)}{\partial\mu^{n+1}} \quad \forall(\mu,\sigma) \in \mathbf{M}. \tag{12}$$

Proof: Chipman (1973) shows that X being normally distributed implies $U_\sigma = \sigma U_{\mu\mu}$ for all (μ,σ), which corresponds to the RHS of (12) for $n = 1$. Further differentiation with respect to μ yields the RHS of (12) for all integers $n > 1$. Now suppose that the RHS of (12) holds for some arbitrary $n \in \mathbb{N}$ (and thus for all larger n). Transform this differential equation as follows ($f(x) = F'(x)$):

$$\begin{aligned}
& \int_{-\infty}^{\infty} x u^{(n)}(\mu+\sigma x)\mathrm{d}F(x) = \sigma \int_{-\infty}^{\infty} u^{(n+1)}(\mu+\sigma x)\mathrm{d}F(x) \\
\iff \quad & -\sigma \int_{-\infty}^{\infty} \left(u^{(n+1)}(\mu+\sigma x) \int_{-\infty}^{x} z f(z)\mathrm{d}z \right) \mathrm{d}x = \sigma \int_{-\infty}^{\infty} u^{(n+1)}(\mu+\sigma x) f(x)\mathrm{d}x \\
\iff \quad & -\int_{-\infty}^{x} z f(z)\mathrm{d}z = f(x) \quad \forall x \\
\implies \quad & -x f(x) = f'(x) \quad \forall x.
\end{aligned}$$

The first equivalence comes from integration by parts, using $\mathbf{E}x = 0$. The final line follows from differentiating the previous one. Together with the condition that f be a density, it implies that f is the Gaussian function $f(x) = (2\pi)^{(-1/2)} \exp(-x^2/2)$. ■

Proposition 2 has interesting implications both for comparative static analysis and for the analysis of risk attitudes in general.

Remark 8. Eichner and Wagener (2003b, Proposition 1(b)) show that when (μ,σ)-preferences exhibit DARA in the sense of $\partial\beta_1(\mu,\sigma)/\partial\mu < 0$,

$$\frac{\partial U(\mu,\sigma)}{\partial\sigma} - \sigma \cdot \frac{\partial^2 U(\mu,\sigma)}{\partial\sigma^2} > 0 \tag{13}$$

for all (μ,σ) is a necessary condition such that agents reduce their risk-taking in response to the addition or increase of an independent background risk to their choice problem; this condition applies

[7] While the results mentioned only refer to absolute measures of risk attitudes β_n, also the relative and partial relative measures and their properties can be usefully employed in comparative-static analyses. See, e.g., Eichner and Wagener (2004).

[8] Chipman (1973, Theorem 1) presents a (mild) regularity condition on $u(y)$ such that a function $U(\mu,\sigma)$ exists that represents u in the case of normal distributions. We assume this condition to hold.

for all probability distributions and irrespectively of compatibility between EU- and (μ, σ)-framework. In case of the normal distribution, Proposition 2 can be applied to rewrite (13) as:

$$\frac{\partial U}{\partial \sigma} - \sigma \cdot \frac{\partial^2 U}{\partial \sigma^2} = -\sigma^2 \cdot \frac{\partial^3 U}{\partial \mu^2 \partial \sigma} = -\sigma^3 \cdot \frac{\partial^4 U}{\partial \mu^4} > 0. \tag{14}$$

The first of these equalities comes from replacing $\partial U/\partial \sigma$ by $\sigma \cdot \partial^2 U/\partial \mu^2$ from (12) for $n = 1$, and by observing that $\partial^2 U/\partial \sigma^2 = \partial^2 U/\partial \mu^2 + \sigma \cdot \partial^3 U/\partial \mu^2 \partial \sigma$ from differentiating (12) for $n = 1$ with respect to σ. The second equality is then obtained by applying (12) for $n = 3$.
For cases where (μ, σ)- and EU-approach are compatible, we established in Proposition 1 that the condition $\partial^4 U/\partial \mu^4 < 0$ in (14) is tantamount to absolute temperance ($u^{(4)} < 0$) which Eeckhoudt et al. (1996) and Gollier and Pratt (1996) identified to be a prerequisite for independent background risks to exert a tempering effect on risk taking in the EU-framework. As argued in Eichner and Wagener (2003b, Proposition 3), the multivariate normal is the only probability distribution such that EU- and (μ, σ)-approach are compatible in settings with multiple but independent risks. Hence, the equivalence between (13) and absolute temperance, as established for normal distributions via Propositions 1 and 2, can be traced back to identical ideas about plausible comparative statics (namely, that the exposure to additional independent risks makes individuals behave in a more risk-averse manner).

Remark 9. To solicit further implications of Proposition 2, let us define another class of measures for risk attitudes by[9]

$$\tilde{\beta}_n(\mu, \sigma) := -\frac{\partial^{n+1} U(\mu, \sigma)}{\partial \mu^{n+1}} \Big/ \frac{\partial^n U(\mu, \sigma)}{\partial \mu^n} \tag{15}$$

for $n \in \mathbb{N}$. Propositions 1 and 2 imply that $\beta_n(\mu, \sigma) = \sigma \cdot \tilde{\beta}_n(\mu, \sigma)$ iff $X \sim N(0, 1)$. Clearly, β_n and $\tilde{\beta}_n$ then also possess identical monotonicity properties with respect to μ. From Corollary 1 we obtain

Corollary 3. *Suppose that $B_n(y) > 0$ for all $y \in \mathbf{Y}$. Then:*

$$\Big[B_n(y) \le B_{n+1}(y)\ \forall y \in \mathbf{Y} \iff \tilde{\beta}_n(\mu, \sigma) \le \tilde{\beta}_{n+1}(\mu, \sigma)\ \forall (\mu, \sigma) \in \mathbf{M}\Big] \iff X \sim N(0, 1). \tag{16}$$

While this might more or less look like a technical nicety, Corollary 3 gets greater significance when seen against the background of the following result:

Proposition 3. *(Gollier, 2000, Proposition 23 and p. 116) Suppose that $B_n(y) > 0$ for all $y \in \mathbf{Y}$. Then*

$$B_n(y) \le B_{n+1}(y) \quad \forall y \in \mathbf{Y} \implies \tilde{\beta}_n(\mu, \sigma) \le \tilde{\beta}_{n+1}(\mu, \sigma) \quad \forall (\mu, \sigma) \in \mathbf{M}, \tag{17}$$

while the converse is generally not true.

Gollier (2000) originally phrases this result in terms of *indirect* (or *derived*) utility functions à la Kihlstrom et al. (1981). Since the function $U(\mu, \sigma)$ formally represents such an indirect utility function,[10] the presentation of Proposition 3 in (μ, σ)-terms is indeed admissible. Corollary 3 then immediately implies that the converse of (17) only holds in the case of Gaussian stochastics.

[9] Functions such as $\tilde{\beta}_n$ are generally inappropriate measures for risk attitudes since they are not immune to arbitrary monotonic transformations of U. However, if underlying U is a cardinal vNM-utility index u, only positively affine transformations of U are admissible. In such a case, measures (15) might indeed make sense.

[10] Given a vNM index $u(y)$ and a random variable z with zero expectation, indirect (or derived) utility is defined as $v(y) := \mathbf{E}_z u(y + z)$. Putting $y = \mu$ and $x = z/\sigma$, we get that $U(\mu, \sigma) = \mathbf{E}_x u(\mu + \sigma x)$ is an indirect utility function.

Proposition 3 has several implications (see Gollier, 2000, pp. 115f). E.g., Proposition 3 implies that DARA is preserved by the addition of an independent background risk. I.e., if an agent becomes less risk averse upon an increase in non-random wealth, he will exhibit that preference pattern in the presence of additional background uncertainty, too (Gollier, 2000, Corollary 3). However, since the converse of Proposition 3 does not generally hold, we cannot infer from an agent behaving in a decreasingly risk averse manner in the presence of background risks that he exhibits the same behavioural trait without background risks, too. Corollary 3 demonstrates that such a conclusion is valid only if stochastics are normally distributed.[11]

6. Conclusion

Our paper emphasizes strong linkages between the EU- and the two-parameter approach. While such linkages do not come as entirely surprising in the location/scale-framework, it is still a different task to pin them down precisely. We achieve this for a wide class of measures for risk attitudes that also have important applications in comparative static analyses under risk. As the last section reveals, a detour via the two-parameter approach may yield useful new insights in the more general EU-framework, too. Finally, as indicated *en passant*, the formal equivalences established in this paper may be instrumental for the analysis of comparative-static problems of decision making under risk both in the EU- and the two-parameter framework.

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[11] I.e., in the presence of a background risk, *final* wealth must be normally distributed. In particular, this will be the case when both primary and background risk are Gaussian. Clearly, this assumption will still keep us within the location-scale framework.

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