Minimax Payoffs

in Sequential One-Round Location Games

Andreas Wagener[*]

Faculty of Economics and Management

University of Hannover

Koenigsworther Platz 1

30167 Hannover, Germany

Phone/Fax: +49 - 511 - 762 5874/4574

e-mail: wagener@sopo.uni-hannover.de

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Abstract

In a sequential one-round location game, two firms decide one after the other where to locate in a given market area in order to attract as many customers as possible. For various scenarios we establish upper bounds for the first-mover disadvantage which this game entails. The first-mover should always locate in, or as close as possible to, a center-point of the market. In convex market areas of two dimensions he can always capture at least 4/9 of the market. We provide generalizations for markets with higher-dimensionsal characteristics.

JEL-Classification: R30, C72, D43

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1 Introduction

In a competitive location game, a number of players (e.g., firms, sellers, or political parties) non-cooperatively decide where to place themselves in some given space with the aim of attracting as many consumers or voters as possible. "Location" may be understood in geographical terms as the physical place where a firm, consumer, or facility is situated. It can also be understood metaphorically as an agent's position in a political spectrum (*à la* Downs, 1957) or in a space of product characteristics (Eiselt et al., 1993). If players move in a predetermined order the location game is called sequential.

We consider the simplest sequential location game: With the aim of maximizing his market share, each of two players A and B chooses one site (for a single-good shop, say) from a given set of potential locations. Player A moves before Player B. Consumers' locations are invariantly fixed and uniformly distributed over some area. Each consumer patronizes one store and purchases one item of the good from the seller who is located closest to him or her.[1] Prices are set exogenously and identical at each store.

If the set of possible locations is a closed interval over which consumers' locations are uniformly distributed, this problem coincides with Hotelling (1929)'s classical location problem (without decisions on pricing). In its equilibrium — which involves identical locational choices both in the simultaneous and in the sequential version of the game — both players settle in the midpoint of the interval and share the market equally (see Example 1 in Prescott and Visscher, 1977).[2] When departing from the original Hotelling set-up (i.e., when giving up any of the assumptions two players, locations to be chosen on a finite line, and uniform consumer distribution), even simple sequential location games become quite involved. Only little is known about how the game should be played (Ahn et al., 2004; Economides et al., 2004).

Here, we give up Hotelling's assumption that locations can only be chosen on a line. Rather we switch to locations in an (at least two-dimensional) area. Clearly, whenever the space of locations is not centrally symmetric (and both players are allowed to settle on top of each other), Player A suffers a first-mover disadvantage: By simply copying Player A's locational choice, Player B can always capture the same market share (of

50%) as Player A — but Player B can generally do better than this, at the detriment of Player A. The question we try to answer in this paper is: *by how much*? I.e., we are interested in the magnitude of the first-mover disadvantage, measured by the difference between Player B's and Player A's equilibrium payoffs.

A possibly surprising answer to this question is that the first-mover disadvantage is actually not that big. If the space of possible locations is two-dimensional and convex, it amounts to at most 1/9 of the total market area, i.e., Player A can ensure himself at least 4/9 of the market space, and the difference to the equal-sharing of markets is just 1/18 of total market.

We arrive at this result by a purely geometric approach, using results from the theory of centerpoints. The notion of a centerpoint of a set of points (here: locations) in two- or higher-dimensional space generalizes the concept of the median for a (finite) set on the real line. The first mover should locate in a centerpoint of the market area (which is unique in convex-shaped market areas). We then utilize the fact that for every two-dimensional convex figure any line through the centerpoint partitions the figure into two parts each of which covers an area of at least 4/9 of the total area.

Our analysis complements recent papers by Chawla et al. (2003, 2006). These studies establish 1/3 as an upper bound for the first-mover disadvantage in a two-dimensional one-round location game (generally, it amounts to $(d-1)/(d+1)$ for d-dimenional markets). This bound applies whenever the number of possible locations is finite. We establish the same bound for sequential location games where the set of potential locations is compact but infinite and at least covers the full support of the consumers' locational distribution. Strengthening the requirements that market areas are convex then narrows down the first-mover disadvantage to at most 1/9. We provide generalized upper bounds for market areas of more than two dimensions.

The sequential Hotelling game (without a price subgame) was discussed by Prescott and Visscher (1977) and Rothschild (1979, 1976), even for more than two firms. However, they only consider one-dimensional markets (i.e., the set of possible locations is an interval), where the two-player case is trivial due to symmetry. Adding a price-subgame does not alter this in the case of two firms (Lane, 1980).[3] Here, we analyse non-symmetric markets (for two entrants) – which then also results in an unequal split of market shares. Moreover, our focus is not so much on locational choices but on the (relative) payoffs which are associated with them.

The rest of the paper is organized as follows: Section 2 sets up the framework. Going through various scenarios for market shape, Section 3 then provides the results: for compact market shapes (Player A gets at least a market share of 1/3), triangular markets (Player A a market share of 4/9), and for convex and bounded markets (Player A gets at least 4/9). Section 4 briefly concludes.

2 The Model

Let $C \subset \mathbb{R}^d$ with $d \geq 1$ be a compact set over which consumers' locations are distributed uniformly;[4] we will call C the *market area*. In large parts of this paper, we assume that $d = 2$. The total mass of consumers is normalized to unity. Two sellers, Player A and Player B, each choose where to locate in $L \subset \mathbb{R}^d$. In general, the set of potential locations L for sales outlets and the market area C need not overlap, but we will most often (and naturally) assume that the set of possible locations at least fully covers the market area (i.e., $C \subseteq L$).

Players move sequentially, starting with Player A. After locations have been chosen, each consumer buys one unit from the seller whose location is closest to him or her. Closeness is measured by Euclidean distance. If both sellers' locations are at equal distance to his or her location, a consumer randomizes with probability 1/2 over the locations for his or her purchase. Each seller aims at maximizing his (expected) market share.

With locations $a, b \in L$ chosen by Players A and B and with $\delta(x, y)$ denoting the Euclidean distance between x and y (both in $\mathbb{R}^d$),

$$\begin{aligned}
\mathcal{A}(a,b) &:= \{x \in C : \delta(x,a) < \delta(x,b)\}, \\
\mathcal{B}(a,b) &:= \{x \in C : \delta(x,b) < \delta(x,a)\}, \\
\mathcal{N}(a,b) &:= \{x \in C : \delta(x,a) = \delta(x,b)\}
\end{aligned}$$

measure, respectively, the fraction of consumers who buy from Player A, buy from Player

B, or randomize between them. Hence, payoffs π_i of of Players $i = A, B$ are given by

$$\pi_A(a,b) = \int_{\mathcal{A}(a,b)} \mathrm{d}x + \frac{1}{2} \cdot \int_{\mathcal{N}(a,b)} \mathrm{d}x \quad \text{and} \quad \pi_B(a,b) = \int_{\mathcal{B}(a,b)} \mathrm{d}x + \frac{1}{2} \cdot \int_{\mathcal{N}(a,b)} \mathrm{d}x.$$

Clearly, $\pi_A(a,b) + \pi_B(a,b) = 1$ for all $(a,b) \in L^2$. Moreover, as $\pi_i(x,x) = 0.5$ for $i = A, B$ and all $x \in L$ and as Player B moves second, he will never do worse than getting half of the market (provided he behaves rationally, of course). Hence, Player A faces a first-mover disadvantage.

For Player A's locational choice $a \in \mathbb{R}^d$,

$$b(a) := \arg\max_{b \in L} \pi_B(a,b)$$

is Player B's best-reply correspondence. Player A's payoff in the (subgame-perfect) Nash equilibrium of the sequential location game then amounts to

$$\pi_A^* = \max_{a \in L} \left\{ \pi_A(a,b) : b \in b(a) \right\}.$$

Since the game is constant-sum, Player A's equilibrium payoff equals his minimax payoff:

$$\pi_A^* = m_A := \max_{a \in L} \min_{b \in L} \pi_A(a,b).$$

Player B's equilibrium payoff amounts to $1 - m_A$ such that the first-mover disadvantage, defined as the difference between the payoffs of the two players, amounts to

$$D = 1 - 2 \cdot m_A.$$

Chawla et al. (2003) have recently shown that with a finite set of possible locations in d-dimensional space the first-mover disadvantage is never larger than $(d-1)/(d+1)$:

Theorem 1 (Chawla et al., 2003) *Suppose that L is a finite set of locations. Player A's payoff in the equilibrium of the d-dimensional sequential location game is at least $1/(d+1)$ and at most $1/2$: For all $C \subset \mathbb{R}^d$ and all finite $L \subset \mathbb{R}^d$,*

$$\frac{1}{1+d} \leq m_A \leq \frac{1}{2}.$$

The equilibrium strategy for Player A consists in choosing that point in L that is closest in Euclidean distance to the set of centerpoints of C. A point $a^* \in \mathbb{R}^d$ is called a *centerpoint* of a mass distribution in $\mathbb{R}^d$ if every closed halfspace $H \subset \mathbb{R}^d$ that contains a^* has mass of at least $1/(d+1)$. Specifically, if C is a finite set of n points in $\mathbb{R}^d$, then a^* is a centerpoint of C if each closed half-space containing a^* contains at least $n/(d+1)$ points of C (observe that a need not be in C). As a consequence of Helly's Theorem (see, e.g., Berger, 1987), each finite set of points in $\mathbb{R}^d$ has at least one centerpoint.[5]

In what follows, we provide a couple of variants for Theorem 1. For some scenarios, this leads to a considerable strengthening of the lower bound for the first-mover's maximin payoff.

3 Results

3.1 Infinitely many locations in compact market areas

We will first consider the case of two-dimensional market areas. Unlike Chawla et al. (2003) we assume that possible locations include at least the convex hull of the market area,[6]

$$\operatorname{conv}(C) \subseteq L.$$

Hence, there is an infinity of potential locations. Nevertheless, the result below is very much in line with Chawla et al. (2003)'s finding. It also relies on the notion of centerpoints. If C is a compact (i.e., bounded and closed) subset of $\mathbb{R}^d$, then any hyperplane through a centerpoint of C partitions C into two pieces the smallest of which has mass of at least $1/(d+1)$. Moreover, if C is a convex planar set, the centerpoint of C is the (unique) point a^* that maximizes the cut-off area function

$$\phi(a) := \min \left\{ \int_{H \cap C} \mathrm{d}x \,\middle|\, H \text{ is a halfplane that contains } a \right\}. \tag{1}$$

For any bounded (but not necessarily convex) area, a centerpoint exists. I.e., for $d = 2$, any straight line through the centerpoint a of C devides C into parts whose mass is at least $1/3$ each. An instructive geometric proof of this theorem can be found in Yaglom and Boltjanskiĭ (1961, Theorem 2.6c (p. 19, pp. 126f)).

Proposition 1 *Suppose that the market area $C \subset \mathbb{R}^2$ is bounded and that* $\text{conv}(C) \subseteq L$*. Then Player A's minimax payoff in the two-dimensional sequential location game is at least* $1/3$ *and at most* $1/2$*:*

$$\frac{1}{3} \leq m_A \leq \frac{1}{2}.$$

Obvious examples where $m_A = 1/2$ in Proposition 1 are markets of centrally symmetric shape. Here Player A can induce an equal sharing of the market by locating in the center of symmetry (which is the centerpoint); Player B will simply mimic that. In particular, Hotelling (1929)'s classical location problem can be understood as the special case of this where $d = 1$ and $C = L = [0, 1]$.

The worst case for Player A from Proposition 1 is depicted in Figure 1. It represents a market where $C = \{x_1, x_2, x_3\}$ with the mass 1/3 of the consumers in each point, and $L = \mathbb{R}^2$.

Figure 1 goes here.

Player A's optimal strategy is to settle in an arbitrary point a of the triangle $\text{conv}(\{x_1, x_2, x_3\})$. Each of these points is a centerpoint of $C = \{x_1, x_2, x_3\}$ as every line through it partitions C into two subsets each of which has a size of at most $d/(d + 1) = 2/3$. Hence, Player B can maximally chop off 2/3 of the market by appropriately responding to a. One among the infinitely many of such responses is indicated by point b in Figure 1; the corresponding boundary between Player A's and Player B's market is given by the solid line, the bisector of a and b.[7,8] If Player A settles outside the triangle $\text{conv}(\{x_1, x_2, x_3\})$, then Player B will, by locating "between" Player A and the triangle, grab the entire market area. Hence, a market share of 1/3 is the best Player A can achieve.

The problem depicted in Figure 1 constitutes the limitimg case of Proposition 1. The formal **proof** of Proposition 1 is omitted since it is similar to that in Chawla et al. (2003). Summarizing: with infinitely many locations in a compact market area, 1/3 constitutes a lower bound for the first-mover's market share.

3.2 Triangular market areas

We next consider the sequential location game for triangular-shaped market areas. As we shall argue later, this establishes the limiting case of Proposition 3 to be derived below.

Proposition 2 *Suppose that the market area C is a (non-degenerate and bounded) triangle in $\mathbb{R}^2$ and that $C \subset L$. Player A's minimax payoff in the two-dimensional sequential location game is $m_A = 4/9$. Player A chooses the centerpoint of C as his location.*

Proof: Without loss of generality choose C to be an equilateral triangle; by an appropriate affine transformation any triangle can be mapped into an equilateral one.[9] Figure 2 depicts an equilateral triangle $C = \text{conv}(\{x_1, x_2, x_3\})$:

Figure 2 goes here.

In a triangle, medians are concurrent in the centerpoint and are each divided by the centerpoint in a ratio of $2/3 : 1/3$, as measured from the vertices. Hence, cutting through the centerpoint and parallel to a vertex of the triangle cuts off a smaller triangle from the original triangle whose area is $4/9$ of the full triangle.

Suppose that Player A has chosen the centerpoint a of C as his location. Any choice b of Player B yields a bisector of a and b as the boundary between the players' market areas. Obviously, by moving b closer to a (in the sense that the bisector between b and a moves, in a parallel way, closer to a), Player B can increase his market share. Essentially, Player B has to find that line through a that cuts off as large a share of the market as possible. By reasons of symmetry, any such cut can be executed as indicated by line ℓ, hitting the vertices of the market at r and q. By cutting along the median (line ℓ' through x_2 and t, the midpoint of x_1x_3), the market will be shared equally. Now rotate ℓ' downwards to ℓ, say. This will increase the fraction of the market that is below the cut by the area of the quadrilateral $rtx_2'q'$, where x_2' and q' are the symmetric images of x_2 and q with respect to a. The quadrilateral $rtx_2'q'$ has the same area as the difference between the areas ax_2q and rat. Extending this argument, the market share below the cut will then be maximized by the line ℓ'' that runs parallel to x_1x_2 through a. The area of the trapezoid x_1x_2pp' equals, by construction, 5/9 of the area of the triangle (the triangle above pp' contains 4/9 of the area of $x_1x_2x_3$ since both its base line and its altitude are, in length, 2/3 of the corresponding lines in the original triangle).

To see why it is indeed Player A's optimal strategy to locate in the centerpoint a, note first that it can trivially never be optimal for A to locate *outside* C.[10] Next assume that A locates inside C but not at its centerpoint. This offers Player B the opportunity to divide the market parallel to that side of the triangle that has the largest distance from Player A's location. Player B would gather a trapezoid which has, when compared to trapezoids that can be cut off by lines through the centerpoint a, the same base line $x_1x_2 = x_2x_3 = x_1x_3$, but a greater altitude than x_1x_2pp' — and consequently a larger size. Evidently, this would harm Player A and can, thus, not be optimal. •

3.3 Bounded and convex market areas

The following result shows that Proposition 1 can be considerably sharpened if market areas are restricted to be convex:

Proposition 3 *Suppose that $C \subset \mathbb{R}^2$ is a bounded and convex set. Moreover, assume that $C \subset L$. Player A's minimax payoff in the two-dimensional sequential location game is at least* 4/9 *and at most* 1/2*:*

$$\frac{4}{9} \leq m_A \leq \frac{1}{2}.$$

The proof follows from a nice geometric result attributed to Winternitz by Yaglom and Boltjanskiĭ (1961, p. 36). This result shows that the triangular market area discussed in the previous section and its attending minimax payoff of 4/9 indeed constitute a limiting case for all bounded and convex figures:[11]

Theorem 2 (Winternitz) *For any bounded and convex planar body C with unit area and any halfplane H that contains the centerpoint of C, the area of $H \cap C$ is at least* 4/9.

The proof of this theorem — a rigorous version of which is in Yaglom and Boltjanskiĭ (1961, pp. 160-162) — can be outlined with the help of Figure 3:

Figure 3 goes here.

Suppose the market land is given by the boldly surrounded area C; clearly this is bounded and convex. Suppose further that the centerpoint of C is a. Consider any cut through a; for example consider the vertical cut indicated by the line ℓ. Any such cut divides the market area into two parts; given our vertical cut, we call them the left-hand and the right-hand part of C. The dividing line ℓ cuts the boundaries of C at (precisely) two points; call them y and z. Replace now the left-hand part of C by a triangle with vertices x_1, y, and z, such that (i) y and z are the points of intersection of ℓ with the boundary of C and (ii) the triangle has an area equal to that of the left-hand part of C. Next replace the right-hand part of C by a trapezoid yx_2x_3z such that (i) the line segment yz is the smaller base, (ii) points x_2 and x_3 lie on the straight lines x_1y and x_1z, respectively, and (iii) the area of the trapezoid equals that of the right-hand part of C. By construction, we now obtain a triangle conv$\{x_1x_2x_3\}$ that has the same size as the market land C.

Consider now the centerpoint of the triangle conv$(\{x_1x_2x_3\})$; call it a'. It can be shown (actually, that is the crucial step) that a' lies in the *left-hand* part of the original market area C. Now divide the triangle into two pieces (a smaller triangle and a trapezoid) by a cut through a' that runs to the left and parallel to ℓ (and, thus, also to x_2x_3). In our case, this will result in a vertical line ℓ'. We know from Proposition 2 that this cut divides the area of the large triangle conv$(\{x_1x_2x_3\})$ into two parts with proportions 4/9 to 5/9. Also the original cut ℓ partitions the triangle conv$(\{x_1x_2x_3\})$. As ℓ runs to the right of ℓ', the proportion of the areas of the left-hand part conv$(\{x_1yz\})$ and of the right-hand part conv$(\{yx_2x_3z\})$ cannot be larger than $4/9 : 5/9$. Given that these areas are, by construction, identical to those left over by the division ℓ of the original market area C, we have established that any division through a centerpoint of C leaves the smaller area with a size of at least 4/9 of the total market area.

From Proposition 3, the first-mover disadvantage in the sequential location game can be narrowed down to a mere $D = 1/9$ in the case of convex market areas. This bound is surprisingly general — and surprisingly small, both in absolute terms and relative to the value of 1/3 established in Chawla et al. (2003) and Section 3.1.

3.4 Higher dimensions

In a purely spatial interpretation of our set-up, only two dimensional market areas are relevant. As mentioned in the introduction, the theory of locational choice may be also used metaphorically in policy or attribute spaces. Here, the dimensionality is not clear, but often larger than two. Fortunately, Winternitz' Theorem was generalized to an arbitrary fixed number of dimensions by Grünbaum (1960):

Theorem 3 (Grünbaum, 1960) *Let K be a bounded and convex body in Euclidean d-space with a unit volume (measure). Then any hyperplane through the centerpoint of K partitions K into two subsets K_1 and K_2 whose volumes $V(K_i)$ satisfy*[12]

$$V(K_i) \geq \left(\frac{d}{d+1}\right)^d \quad (i = 1, 2).$$

This result can readily be used to extend Proposition 3:

Proposition 4 *Suppose that $C \subset \mathbb{R}^d$ is a bounded and convex set. Moreover, assume that $C \subset L$. Player A's minimax payoff in the d-dimensional sequential location game is at least $[d/(1+d)]^d$ and at most $1/2$:*

$$\left(\frac{d}{1+d}\right)^d \leq m_A \leq \frac{1}{2}.$$

Player A can realize this payoff by locating in the centerpoint of C.

Since $[d/(1+d)]^d > 1/(1+d)$ for all $d > 1$, the lower bound for Player A's market share is higher than in Chawla et al. (2003)'s theorem for finite sets of locations. Moreover, as $((d+1)/d)^d < \mathrm{e}$ for all d (with e as Euler's number), Player A can secure himself at least a market share of $\mathrm{e}^{-1} \simeq 0.368$ in market spaces of arbitrary dimensions. This is in marked contrast to Chawla et al. (2003) where the first-mover's market share potentially approaches zero in high-dimensional markets.

4 Conclusion

We provide upper bounds of the first-mover disadvantage in sequential one-round location games for various scenarios. For convex market areas of dimension d, this disadvantage amounts to a fraction of at most $D = 1 - 2(d/(d+1))^d$ of the market, which ranges between $1/9$ for $d = 2$ and $1 - 2\mathrm{e}^{-1} \simeq 0.264$ for large dimensions. The disadvantage is larger the higher is market dimensionality. The optimal strategy for the first-moving Player A is to always locate in the centerpoint of the distribution of consumers, and the second-moving Player B will settle close to him. In that sense, Hotelling's intuition that firms should locate "in the middle" of the market goes through.

An obvious question for further research is the derivation of minimax payoffs and strategies in location games with several rounds. Apart from some special cases (see, e.g., Ahn et al., 2004; Chawla et al., 2003) not much is known so far about these games. Potentially, the geometric route taken in this paper also yields insights for this more advanced question.

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Endnotes

[1]If the set of locations is continuous and consumers are uniformly distributed over some compact set, sequential location games are also called Voronoi games; we are, thus, considering a one-round Voronoi game (see Cheong et al., 2004).

[2]This assumes that firms can locate on top of one another. If that is not feasible, the first-mover settles in the midpoint of the interval and the other firm locates at an epsilon distance from the midpoint.

[3]Economides et al. (2004) show, however, that with a price subgame the ordering of profits follows the ordering of entries.

[4]Unlike in Hotelling's model with a location and a price subgame (see Anderson and Goeree, 1997), the uniformity assumption can be easily relaxed.

[5]The same holds for continuous mass distributions over $\mathbb{R}^d$ as well as for arbitrary Borel probability measures on $\mathbb{R}^d$.

[6]The convex hull for a set C of points in a real vector space is the minimal convex set containing C; we denote it as conv$C$. Recall that for $x_1, x_2, x_3 \in \mathbb{R}^2$, the set $\text{conv}(\{x_1, x_2, x_3\})$ gives the triangle with vertices x_1, x_2, and x_3.

[7]Given a metric δ, the *bisector* of any two distinct points $a, b \in \mathbb{R}^d$ consists of all points $x \in \mathbb{R}^d$ such that $\delta(a, x) = \delta(b, x)$. With δ as the Euclidean metric, the bisector of a and b is a hyperplane that is normal to the line $\overline{ab}$ and passes through its midpoint.

[8]Observe that the argument also goes through if Player A locates at one of the vertices of $\text{conv}(\{x_1, x_2, x_3\})$.

[9]In particular, the midpoints of the sides, the medians, and the centerpoint of the original triangle will be mapped into the midpoints, the medians, and the centerpoint of the equilateral triangle. All area ratios will remain unchanged.

[10]By the hyperplane separation theorem, any triangle can be strictly separated from any point outside it. This means that $\pi_A = 0$ as Player B could capture the whole market.

[11]In terms of the cut-off area function (1), one can phrase Winternitz' Theorem as follows: *For any bounded and convex planar body C, $\phi(a^*) = 4/9$ where a^* is the centerpoint of C.*

[12]Equality holds here if, and only if, K is a cone with the hyperplane being parallel to the base and K_i is that subset of K that contains the base.

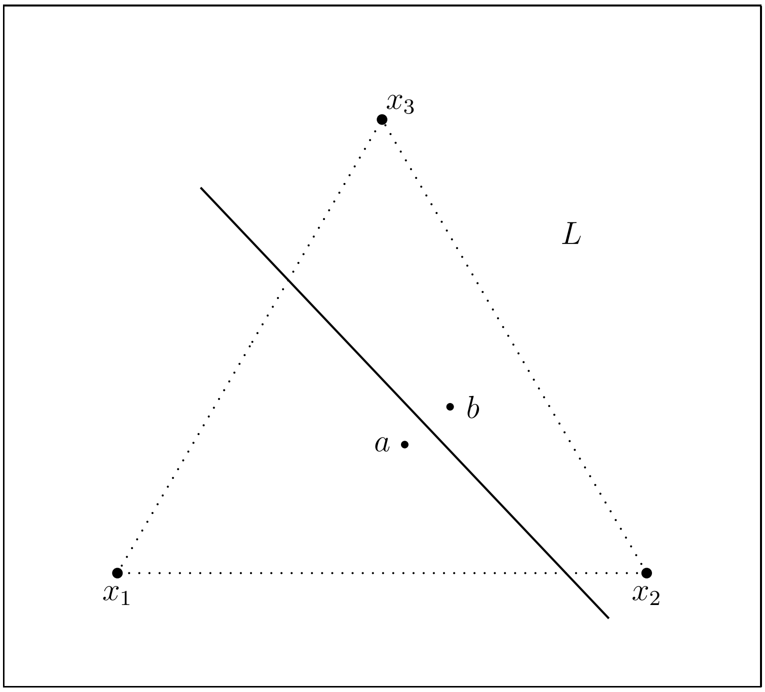

Figure 1: $C = \{x_1, x_2, x_3\}$. Player A gets 1/3 of the market.

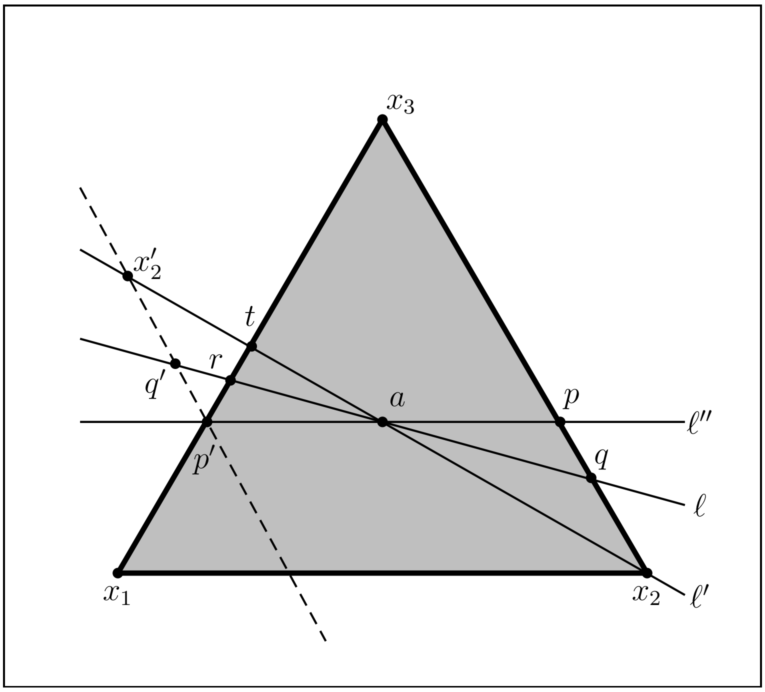

Figure 2: $C = \text{conv}\{x_1, x_2, x_3\}$: Player A gets 4/9 of the market.

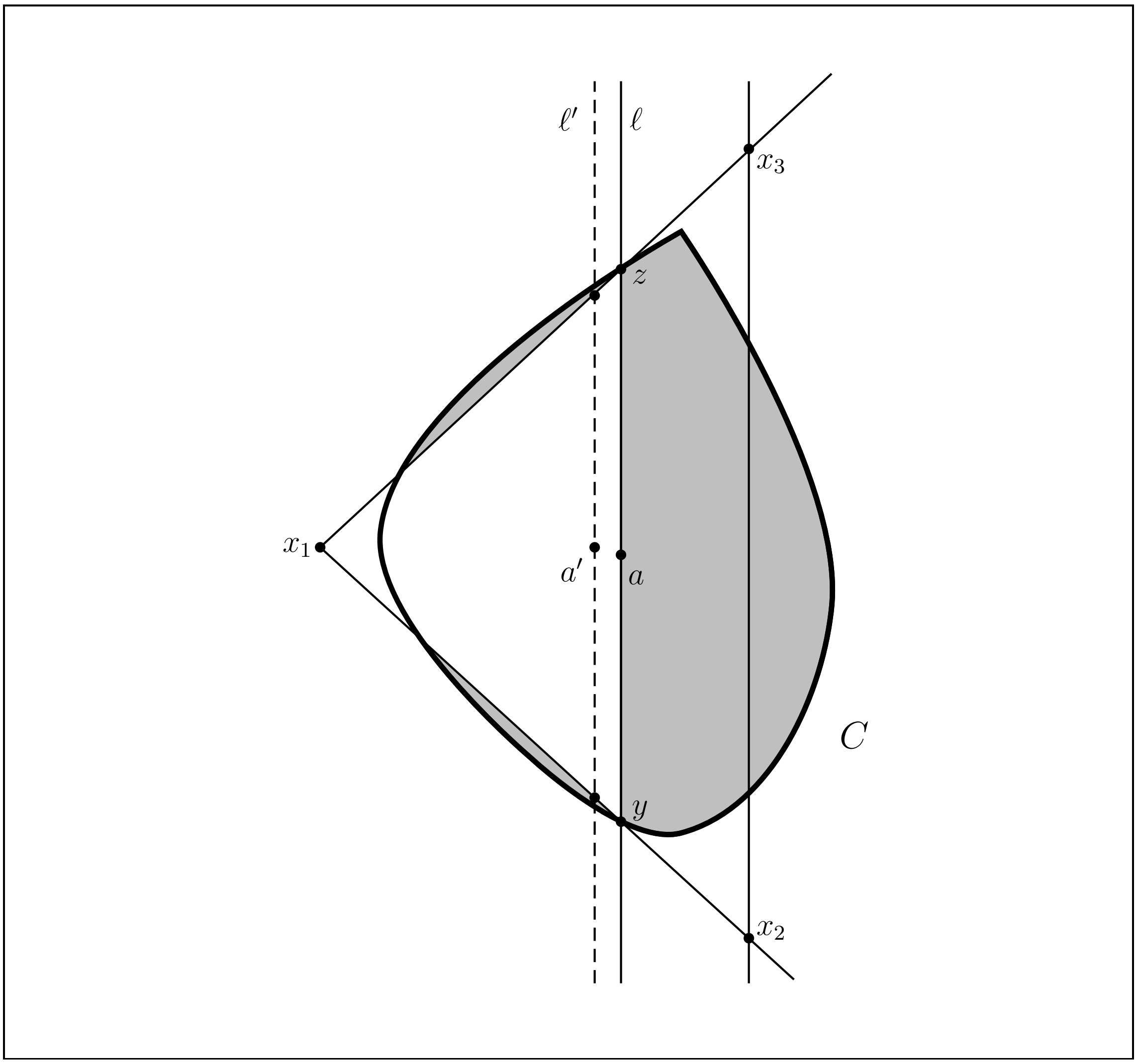

Figure 3: Illustration of Winternitz' result.